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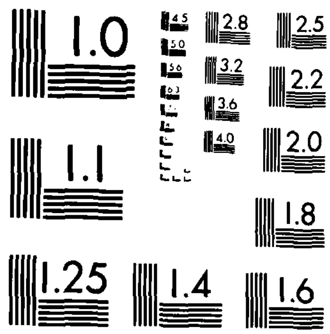

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THE THEORY OF STANDARDIZED TIME SERIES

by

Peter W. Glynn and Donald L. Iglehart

TECHNICAL REPORT NO. 8

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Report Number 8: THE THEORY OF STANDARDIZED TIME SERIES;
by Peter W. Glynn and Donald L. Iglehart

Building on work of SCHRUBEN, we have developed a general framework for the analysis of standardized time series. Under mild assumptions on the output process (see (3.1)), the method of standardized time series produces asymptotically valid confidence intevals for steady-state parameters. However, these intervals are asymptotically larger (see (5.16)) and more variable (see (5.33)) then those steady-state intervals obtained by a method which consistently estimates the appropriate steady-state variance constant (such as the regenerative method). In this sense, standardized time series confidence intervals are asymptotically less desirable then those constructed by consistent estimation.

These results do not, however, preclude the possibility that standardized times series may be superior in certain small sample context; this remains an area for future work.

A-1

1. INTRODUCTION

Let $\mathbf{Y} = \{Y(t) : t \geq 0\}$ be a real-valued stochastic process representing the output of a simulation. To incorporate stochastic sequences $\{Y_n : n \geq 0\}$ into our framework, we set $Y(t) = Y_{[t]}$, where $[t]$ is the greatest integer less than or equal to t. Frequently, a simulator is interested in estimating steady-state parameters associated with $\mathbf{Y}$. Recently, SCHRUBEN (1983) proposed a new class of procedures, based on standardized time series, for dealing with the steady-state simulation problem. Our goal, in this paper is to generalize the method of standardized time series and to study the structure of such procedures. Section 2 reviews the basic concepts of weak convergence upon which the method of standardized time series is based.

In Section 3 the method of standardized time series is introduced and its basic properties are investigated. Section 4 gives examples of standardized times series, while Section 5 discusses the asymptotic behavior of the method. Section 6 provides a short summary of the major results of this paper.

2. WEAK CONVERGENCE OF STOCHASTIC PROCESSES

Let $\mathbf{X} = \{X_n : n \geq 1\}$ be a sequence of real-valued random variables (RV's). The sequence $\mathbf{X}$ is said to converge **weakly** to a r.v. X (written $X_n \Rightarrow X$ as $n \to \infty$) if

$$P\{X_n \leq x\} \to P\{X \leq x\} , \tag{2.1}$$

as $n \to \infty$, for every x which is a continuity point of $P\{X \leq \cdot\}$. This mode of convergence is frequently used in the study of simulation output analysis algorithms. For example, the central limit theorem (CLT) is a weak convergence statement concerning a sequence of normalized partial sums.

It is well known (see, for example, Theorem 4.4.2 of CHUNG (1974)) that the requirement (2.1) is equivalent to demanding that

$$Ef(X_n) \to Ef(X) , \tag{2.2}$$

as $n \to \infty$, for every bounded, continuous function $f : \mathbb{R} \to \mathbb{R}$.

The method of standardized time series requires that one study weak convergence properties of random elements X_n corresponding to stochastic processes. Since a stochastic process may be regarded as a random function, it is natural to generalize to the case where the X_n's take values in a function space. The precise space that we shall require is $C[0,1]$, the elements of which are continuous functions $x : [0,1] \to \mathbb{R}$; see p. 54-61 of BILLINGSLEY (1968) for a thorough description of this space.

As (2.2) indicates, the notion of weak convergence depends upon defining a suitable class of continuous functions. For $x,y \in C[0,1]$, let

$$\rho(x,y) = \sup\{|x(t) - y(t)| : 0 \leq t \leq 1\} ;$$

$\rho(x,y)$ measures the distance between the two elements x,y.

(2.3) DEFINITION. A function $f : C[0,1] \to \mathbb{R}$ is said to be **continuous** if $f(x_n) \to f(x)$ as $n \to \infty$, whenever $\rho(x_n, x) \to 0$ as $n \to \infty$, where the x_n's and x are elements of $C[0,1]$.

By analogy with (2.2), we can now define a notion of weak convergence on $C[0,1]$. Let $\{X_n : n \geq 1\}$ be a sequence of random elements taking values in $C[0,1]$ (in other words, the X_n's correspond to stochastic processes with sample paths in $C[0,1]$). If X is a random element of $C[0,1]$, then $\{X_n : n \geq 1\}$ is said to converge **weakly** to X (written $X_n \Rightarrow X$) if

$$Ef(X_n) \to Ef(X) \tag{2.4}$$

as $n \to \infty$, for every bounded, continuous function $f : C[0,1] \to \mathbb{R}$.

The method of standardized time series is based on the following important result, known as the continuous mapping theorem (CMT). For a (measurable) function $h : C[0,1] \to \mathbb{R}$, let $D(h)$ be the set of elements $x \in C[0,1]$ at which h is discontinuous (in other words, $x \in D(h)$ if there exists a sequence $\{x_n\} \in C[0,1]$ for which $\rho(x_n, x) \to 0$ as $n \to \infty$, with $f(x_n) \not\to f(x)$).

(2.5) PROPOSITION. Suppose X_n, X are random elements of $C[0,1]$ such that $X_n \Rightarrow X$ as $n \to \infty$. If $P\{X \in D(h)\} = 0$, then $h(X_n) \Rightarrow h(X)$ as $n \to \infty$.

See p. 31 of [1] for a proof of this result. Loosely speaking, Proposition 2.5 says that if X_n can be weakly approximated by X, then the real-valued random variable $h(X_n)$ can be weakly approximated, in distribution, by $h(X)$, provided that h is suitably continuous. To be of practical benefit, of course, it is also necessary to choose h's for which the distribution of $h(X)$ is known.

3. STANDARDIZED TIME SERIES

Let $\mathbf{Y} = \{Y(t) : t \geq 0\}$ be a real-valued (measurable) stochastic process representing the output of a simulation. To apply the method of standardized time series to the output process $\mathbf{Y}$, it is necessary to make the following assumption:

(3.1) There exist finite constants μ and σ (σ positive) such that

$$X_n \Rightarrow \sigma B$$

as $n \to \infty$, where B is a standard Brownian motion, and

$$X_n(t) = n^{1/2}(\bar{Y}_n(t) - \mu t)$$

with

$$\bar{Y}_n(t) = \int_0^{nt} Y(s)ds/n \,, \qquad \text{for} \quad 0 \leq t \leq 1 \,.$$

Note that X_n and B are both processes whose sample paths lie in $C[0,1]$, so that the weak convergence required by (3.1) is assumed to take place in the function space $C[0,1]$. A variety of different output processes satisfy (3.1).

(3.2) PROPOSITION. Let Y be a stationary (measurable) ϕ-mixing process (see p. 178 of [1] for a definition) satisfying:

(i) $EY^2(0) < \infty$

(ii) $\int_0^\infty \phi^{1/2}(t)dt < \infty$

(iii) $\int_0^\infty \text{cov}(Y(0), Y(t))dt > 0$.

Then, (3.1) holds with $\mu = EY(0)$ and $\sigma^2 = 2\int_0^\infty \text{cov}(Y(0), Y(t))dt$.

For a proof, see p. 178-179, as well as a remark on p. 150, of [1] (for extensions to non-stationary processes, see p. 179-182 of [1]).

(3.3) PROPOSITION. Let $Y(t) = Y_{[t]}$, where $\{Y_n : n \geq 0\}$ is a strongly mixing, strictly stationary sequence (see HALL and HEYDE (1980) for a definition) satisfying:

(i) $E|Y_0|^{2+\delta} < \infty$

(ii) $\sum_{n=0}^{\infty} \alpha(n)^{\delta/(2+\delta)} < \infty$

(iii) $E(Y_0-r)^2 + 2\sum_{k=1}^{\infty} E(Y_0-r)(Y_k-r) > 0,$

where $r = EY_0$; $\delta > 0$, and $\{\alpha(n) : n \geq 0\}$ is the sequence of mixing constants. Then, (3.1) holds with $\mu = r$ and $\sigma^2 = E(Y_0-r)^2 + 2\sum_{k=1}^{\infty} E(Y_0-r)(Y_k-r)$.

For a proof, see p. 132 of [8].

(3.4) PROPOSITION. Let $Y(t) = Y_{[t]}$, where $\{Y_n : n \geq 0\}$ is an associated sequence of strictly stationary r.v.'s (see NEWMAN and WRIGHT (1981) for definitions) satisfying:

(i) Y_0 is a non-degenerate r.v.

(ii) $\sum_{k=1}^{\infty} E(Y_0-r)(Y_k-r) < \infty$, where $r = EY_0$.

Then, (3.1) holds with $\mu = r$ and $\sigma^2 = E(Y_0-r)^2 + 2\sum_{k=1}^{\infty} E(Y_0-r)(Y_k-r)$. For a proof, see [10].

(3.5) PROPOSITION. Let $\mathbf{Y}$ be a (possibly) delayed regenerative process with regeneration times $0 \leq T(0) < T(1) < T(2) < \cdots$. Set

$$Y_n(f) = \int_{T(n-1)}^{T(n)} f(Y(s))ds$$

$$\tau_n = T(n) - T(n-1)$$

and assume that:

(i) $E(Y_1(|f|)^2 + \tau_1^2) < \infty$

(ii) $E(Y_1(f) - r\tau_1)^2 > 0$, where $r = EY_1(f)/E\tau_1$.

Then, (3.1) holds with $\mu = r$ and $\sigma^2 = E(Y_1(f) - r\tau_1)^2/E\tau_1$.

For a proof, see FREEDMAN (1967) (the argument given there for the Markov chain case easily extends to the general regenerative setting).

Let $h : C[0,1] \to \mathbb{R}$ be the mapping defined by $h(x) = x(1)$; in other words, h evaluates x at the point $t = 1$. It is trivially verified that $h(x_n) \to h(x)$ whenever $\rho(x_n,x) \to 0$ as $n \to \infty$. Hence, Proposition 2.5 implies that

$$X_n(1) \Rightarrow \sigma B(1) ,$$

or

$$n^{1/2}(\bar{Y}(n) - \mu) \Rightarrow \sigma B(1) \tag{3.6}$$

as $n \to \infty$. Application of a standard converging-together argument (see p. 93 of CHUNG [4]) yields

(3.7) PROPOSITION. Assumption (3.1) guarantees that

$$\bar{Y}(n) \Rightarrow \mu$$

as $n \to \infty$.

Thus, (3.1) suffices to guarantee that the steady-state estimation problem for $\mathbf{Y}$ makes sense; μ is the steady-state parameter which the simulator wishes to estimate.

Note that the CLT (3.6) could be used to obtain confidence intervals for μ, provided that σ were known. As Schruben points out in [12], the principle underlying standardized time series is to "cancel out" the σ.

The cancellation procedure involves choosing a function g from the class $\mathcal{M}$; $\mathcal{M}$ is the class of (measurable) functions $g : C[0,1] \to \mathbb{R}$ such that:

(3.8) (i) $g(\alpha x) = \alpha g(x)$ for $\alpha > 0$, $x \in C[0,1]$,

(ii) $g(x - \beta k) = g(x)$ for $\beta \in \mathbb{R}$ and $x \in C[0,1]$, where

$k(t) = t$,

(iii) $P\{g(B) > 0\} = 1$,

(iv) $P\{B \in D(g)\} = 0$.

(3.9) THEOREM. Suppose that $g \in \mathcal{M}$. Under Assumption (3.1),

$$\frac{\bar{Y}_n(1) - \mu}{g(\bar{Y}_n)} \Rightarrow \frac{B(1)}{g(B)} \qquad \text{as } n \to \infty . \tag{3.10}$$

PROOF. Let $h : C[0,1] \to \mathbb{R}$ be the mapping defined by $h(x) = x(1)/g(x)$ for $g(x) \neq 0$ (and zero elsewhere). Assumptions (3.8iii) and (3.8iv) allow one to verify that $P\{\sigma B \in D(h)\} = 0$. Thus, Proposition 2.5 guarantees that

$$h(X_n) \Rightarrow h(\sigma B)$$

as $n \to \infty$. By (3.8i), $h(\sigma B) = B(1)/g(B)$ (recall that $\sigma^2 > 0$). Furthermore,

$$\begin{aligned} h(X_n) &= \frac{n^{1/2}(\bar{Y}_n(1) - \mu)}{g(n^{1/2}(\bar{Y}_n - k\mu)} \\ &= \frac{\bar{Y}_n(1) - \mu}{g(\bar{Y}_n - k\mu)} \\ &= \frac{\bar{Y}_n(1) - \mu}{g(\bar{Y}_n)} , \end{aligned}$$

For the second step, observe that the symmetry of H proves that $H(z(g;\ 1-\delta/2)) - H(-z(g;\ 1-\delta/2)) = 1-\delta$. Set $b = z(g;\ 1-\delta/2)$. Then, for any $\alpha \in \mathbb{R}$, (5.5) yields

$$H(\alpha+2b) - H(\alpha) \leq 1-\delta .$$

Thus, in order that $H(\beta) - H(\alpha) = 1-\delta$, it must be that $\beta-\alpha \geq 2b$; proving our assertion.

We turn now to the choice of $g \in \mathcal{M}$. Our goal is to find g minimizing

$$\Phi(g) = Eg(B) \cdot z(g;\ 1-\delta/2) . \tag{5.6}$$

Note that the criterion (5.6) is scale-invariant.

(5.7) LEMMA. For $b > 0$, $\Phi(bg) = \Phi(g)$.

PROOF. Note that $z(g;\ 1-\delta/2)$ solves

$$\begin{aligned} 1 - \delta/2 &= P\{B(1) \leq z(g;\ 1-\delta/2) \cdot g(B)\} \\ &= P\{B(1) \leq \frac{1}{b} z(g;\ 1-\delta/2) \cdot b \cdot g(B)\} \\ &= P\{B(1) \leq z(bg;\ 1-\delta/2) \cdot b \cdot g(B)\} \end{aligned}$$

so that the continuity and strict monotonicity of H imply that

$$z(gb;\ 1-\delta/2) = \frac{1}{b} z(g;\ 1-\delta/2) .$$

Relation (5.6) then yields the lemma.

Clearly, it is desirable to obtain confidence intervals with as small an expected length as possible. From Proposition 5.1, it seems reasonable to therefore choose α, β, and $g \in \mathscr{M}$ such that $Eg(B) \cdot (\beta-\alpha)$ is minimized.

(5.3) PROPOSITION. Suppose $g \in \mathscr{M}$. Then, for a $100(1-\delta)\%$ confidence interval, $\beta-\alpha$ is minimized by choosing

$$\beta = z(g;\ 1 - \delta/2)$$

$$\alpha = -\beta$$

where $z(g;\ x)$ solves the equation $H(z(g;\ x)) = P\{B(1)/g(B) \leq z(g;\ x)\} = x$ (in other words, the confidence interval should be centered at $\bar{Y}_n(1)$).

PROOF. We proceed in two steps. First, for any $a \in \mathbb{R}$ and $b,y \geq 0$, it is easily verified that

$$\Phi((a+2b)y) - \Phi(ay) \leq \Phi(by) - \Phi(-by) . \tag{5.4}$$

Integrating both sides of (5.4) with respect to $G(dy)$ and using (3.14), we get

$$H(a+2b) - H(a) \leq H(b) - H(-b) . \tag{5.5}$$

Furthermore, the symmetry of Φ and (3.14) implies that H is also symmetric, in the sense that $H(b) - H(0) = H(0) - H(-b)$ for $b \geq 0$.

(5.1) PROPOSITION. Assume $g \in \mathscr{M}$, and that (3.1) holds.

(a) If g is non-negative, then

$$\varliminf_{n \to \infty} n^{1/2} EL_n \geq /Eg(B) \cdot (\beta-\alpha) \quad .$$

(b) If $\{g(X_n) : n \geq 1\}$ is uniformly integrable, then

$$\lim_{n \to \infty} n^{1/2} L_n = \sigma Eg(B) \quad \cdot (\beta-\alpha) \ .$$

Assumption (3.1) and Proposition 2.5 guarantee that if $g \in \mathscr{M}$, then

$$g(X_n) \Rightarrow \sigma g(B) \ , \tag{5.2}$$

as $n \to \infty$. If g is non-negative, then Fatou's lemma can be applied to (5.2) to conclude that

$$Eg(B) \leq \varliminf_{n \to \infty} Eg(X_n) \ ,$$

proving (a). On the other hand, it is well-known (see CHUNG (1974), p. 96) that uniform integrability implies that

$$Eg(B) = \varliminf_{n \to \infty} Eg(X_n) \ ,$$

proving (b).

(4.10) EXAMPLE. Let $b : C[0,1] \to \mathbb{R}$ be defined by

$$b(x) = x(t^*)/(t^*(1-t^*))^{1/2}$$

where $t^* = \inf\{t \geq 0 : x(t^*) = M^*\}$, $M^* = \max\{x(t) : 0 \leq t \leq 1\}$. SCHRUBEN (1982) showed that $(b^2 \circ \Gamma)(B)$ has a chi-square distribution with 3 degrees of freedom. Consequently,

$$\sqrt{m}\ \tilde{g}_m(B) \overset{\mathcal{D}}{=} (\chi^2_{3m})^{1/2}$$

so that

$$\frac{B(1)}{\sqrt{m}\ \tilde{g}_m(B)} \overset{\mathcal{D}}{=} t_{3m} \quad ,$$

where t_{3m} is a Student's-t RV with $3m$ degrees of freedom. Confidence intervals based on $\tilde{g}_m(B)$ as defined above are the **standardized maximum intervals** of [12].

5. ASYMPTOTICS FOR STANDARDIZED CONFIDENCE INTERVALS

In this section, we study certain asymptotic properties of standardized confidence intervals. In particular, we consider the asymptotics of the expected length of such confidence intervals, as well as the end-point variability of these intervals.

Now, from (3.15), it is clear that the width of the interval (3.15) is given by

$$L_n = g(\bar{Y}_n) \cdot (\beta-\alpha) \ .$$

$$\sqrt{12m}\,\tilde{g}_m(B) \overset{\mathcal{D}}{=} \Big(\sum_{i=0}^{m-1} B_i^2(1)\Big)^{1/2} = (\chi_m^2)^{1/2} ,$$

where χ_m^2 denotes a chi-square RV with m degrees of freedom. The chi-square property of $\tilde{g}_m(B)$ makes standardized time series based on $\tilde{g}_m$ particularly attractive, since in that case

$$\frac{B(1)}{(12m)^{1/2}\,\tilde{g}_m(B)} \overset{\mathcal{D}}{=} t_m$$

where t_m is the Student's-t distribution with m degrees of freedom; the limit Theorem (3.10) can then be used to construct confidence intervals for μ. These confidence intervals, which were suggested by SCHRUBEN (1983), are based on the so-called **standardized sum process** $(\bar{Y}_n(1) - \mu)/(12m)^{1/2}$ $\tilde{g}_m(\bar{Y}_n(\cdot))$.

(4.9) EXAMPLE. The map $b : C[0,1] \to \mathbb{R}$ defined by

$$b(x) = \int_0^1 |x(t)|\, dt$$

also lies in the class $\mathcal{N}$. Furthermore, the distribution of $(b\circ\Phi)(B)$ is known; see JOHNSON and KILLEEN (1983). However, the distributions of both $g_m^*(B)$ and $\tilde{g}_m(B)$ are quite complicated, and this would appear to limit the applicability of this method.

To calculate the distribution of $g_m^*(B)$ and $\tilde{g}_m(B)$, it is convenient to first find the distribution of $(b\circ\Gamma)(B)$. Note that the continuity of B implies that

$$(4.8) \qquad \frac{1}{m}\sum_{i=1}^{m} (\Gamma B)(i/m) \to \int_0^1 (\Gamma B)(t)dt$$

as $m \to \infty$, a.s. The left-hand side of (4.8) is normally distributed with mean zero and variance

$$\frac{1}{m^2}\sum_{k=1}^{m}\sum_{\ell=1}^{m} \mathrm{cov}[(\Gamma B)(k/m), (\Gamma B)(\ell/m)]$$

$$= \frac{1}{m^2}\sum_{k=1}^{m}\sum_{\ell=1}^{m} [\min(\frac{k}{m}, \frac{\ell}{m}) - \frac{k\ell}{m^2}] = v(m) .$$

Note that $v(m)$ is a Riemann approximation to the integral

$$\int_0^1\int_0^1 [\min(s,t) - st]ds\, dt$$

which has value $1/12$. Thus, $v(m) \to 1/12$; hence, taking characteristic functions of both sides of (4.8) shows that the right-hand side of (4.8) is normally distributed with mean-zero and variance $1/12$. Since $\Lambda_D B, \ldots, \Lambda_{m-1}B$ are independent Brownian motions, it follows that

$$\sqrt{12m}\, g_m^*(B) \overset{\mathcal{D}}{=} \sum_{i=0}^{m-1} |B_i(1)|$$

where $B_D, \ldots, B_{m-1}$ are independent standard Brownian motions; on the other hand, intervals converge to those associated with a normal approximation. This phenomenon is consistent with that observed in the method of batch means, where it is known that as $m \to \infty$, the Student's-t distribution approaches a normal.

A second "extension" method involves defining the class

$$\mathcal{N}_2 = \{b \in \mathcal{N} : P\{B \in D(b^2 \circ \Gamma)\} = 0\} ,$$

where $b^2(x) = b(x) \cdot b(x)$. The following proposition has a proof similar to that of Proposition 4.2.

(4.5) PROPOSITION. If $b \in \mathcal{N}_2$, then $\tilde{g}_m \in \mathcal{M}(m \geq 1)$, where

$$\tilde{g}_m = \left(\sum_{i=0}^{m-1} b^2 \circ \Gamma \circ \Lambda_i\right)^{1/2} .$$

The analogue to Proposition 4.4 is then given by

(4.6) PROPOSITION. $\tilde{g}_m(B) \Rightarrow (E(b^2 \circ \Gamma \circ B))^{1/2}$ as $m \to \infty$.

Thus, confidence intervals based on $\tilde{g}_m(\cdot)$ will, for large m, correspond to that associated with a normal approximation. We now turn to some specific examples of g_m^*'s and $\tilde{g}_m$'s.

(4.7) EXAMPLE. Let $b : C[0,1] \to \mathbb{R}$ be defined by

$$b(x) = \left|\int_0^1 x(t)dt\right| .$$

(4.4) PROPOSITION. $g_m^*(B)/m^{1/2} \Rightarrow E(b\circ\Gamma\circ B)$ as $m \to \infty$.

PROOF. Note that

$$g_m^*(B)/m^{1/2} = \frac{1}{m}\sum_{i=0}^{m-1} b\circ\Gamma\circ(m^{1/2}\Lambda_i B) .$$

But $\{m^{1/2}\Lambda_i B : 0 \leq i < m\}$ has the same distribution as a collection of m independent standard Brownian motions $\{B_i : 0 \leq i < m\}$, so

$$g^*(B)/m^{1/2} \overset{\mathcal{D}}{=} \frac{1}{m}\sum_{i=0}^{m-1} b\circ\Gamma\circ B_i$$

($\overset{\mathcal{D}}{=}$ denotes equality in distribution). Of course, the strong law of large numbers guarantees that

$$\frac{1}{m}\sum_{i=0}^{m-1} b\circ\Gamma\circ B_i \to E(b\circ\Gamma\circ B) \quad \text{a.s.}$$

as $m \to \infty$, proving the result.

Thus, if $E(b\circ\Gamma\circ B) < \infty$, we observe that

$$\frac{m^{1/2}(\bar{Y}_n(1) - \mu)}{g_m^*(\bar{Y}_n)/E(b\circ\Gamma\circ B)} \Rightarrow \frac{m^{1/2}B(1)}{g_m^*(B)/E(b\circ\Gamma\circ B)}$$

as $n \to \infty$, where the limit RV, for large m, is the normally distributed quantity $B(1)$. Hence, as the method of standardized time series is extended to more and more increments, the corresponding confidence

(4.3)
$$\Psi_i = \Gamma \circ \Lambda_i \ ,$$
$$\Psi_i = \Psi_i \circ \Gamma \ .$$

It is evident from (4.3) that

$$g_m^* = \sum_{i=1}^{m} b \circ \Psi_i = \sum_{i=1}^{m} b \circ \Psi_i \circ \Gamma \ ,$$

so that if

$$b_m = \sum_{i=1}^{m} b \circ \Psi_i \ ,$$

we have a representation of g_m^* of the form $g_m^* = b_m \circ \Gamma$.

Clearly, b_m satisfies (3.11i). For (3.11ii), observe that $\Lambda_i B$ is a Brownian motion so (3.11ii) implies that $(b \circ \Gamma \circ \Lambda_i)(B) > 0$ a.s. for $0 \leq i < m$, thus yielding (3.11ii) for b_m.

For (3.11iii), note that the continuity of Λ_i implies that

$$D(g_m^*) \subset \bigcup_{i=0}^{m-1} \{x : \Lambda_i x \in D(b \circ \Gamma)\}$$

so that

$$P\{B \in D(g_m^*)\} \leq \sum_{i=0}^{m-1} P\{\Lambda_i B \in D(b \circ \Gamma)\} \ .$$

But $\Lambda_i B$ is a Brownian motion so that (3.11iii) shows that $P\{\Lambda_i B \in D(b \circ \Gamma)\} = 0$ for $0 \leq i < m$, yielding (3.11iii) for g_m^*.

It is of some interest to consider the behavior of $g_m^*(B)$ for large m.

As we have already seen, the fundamental assumption of the method of standardized time series is that the output process may be approximated by a Brownian motion. Intuitively, then, it should follows that the increments of the output process can be approximated by the increments of Brownian motion. This suggests that one might try to extend the power of the method of standardized time series by applying the procedure separately to each increment of the output process, and then "patching" the increments together. In some sense, this phenomenon occurs in the method of batch means, and is related to the somewhat arbitrary nature of the parameter m. In any case, we now present two "extension" methods.

Let $\Lambda_i : C[0,1] \to C[0,1]$ be the map defined by

$$(\Lambda_i x)(t) = x((i+t)/m) - x(i/m) , \qquad 0 \leq t \leq 1$$

for $0 \leq i < m$ $(m \geq 1)$; the key to our first "extension" procedure is the following result.

(4.2) PROPOSITION. If $b \in \mathcal{N}$, then $g_m^* \in \mathcal{M}$ $(m \geq 1)$, where

$$g_m^* = \sum_{i=0}^{m-1} b \circ \Gamma \circ \Lambda_i .$$

PROOF. We shall show g_m^* can be represented as $g_m^* = b_m \circ \Gamma$, where $b_m \in \mathcal{N}$, thereby proving that $g_m^* \in \mathcal{M}^*$. Let $\Psi_i : C[0,1] \to C[0,1]$ be given by

$$(\Psi_i x)(t) = x((i+t)/m) - x(i/m)(1-t) - tx((i+1)/m)$$

for $0 \leq t \leq 1$ $(0 \leq i < m)$. The following relations are easily verified:

$$g_m(x) = \left[\left(\frac{m}{m-1}\right) \sum_{i=1}^{m} (\Delta_m\, x(i/m) - x(1)/m)^2\right]^{1/2} ,$$

where $\Delta_h\, x(t) = x(t) - x(t - 1/h)$. Note that $\Delta_m\, B(i/m)$ $(i = 1, \ldots, m)$ are increments of standard Brownian motion, and are therefore independent and identically distributed normal RV's with mean zero and variance $1/m$. Also, $B(1)/m$ is the sample mean of these increments. Hence $B(1)/g_m(B)$ has a Student's-t distribution with $m-1$ degrees of freedom.

On the other hand,

$$g_m(\bar{Y}_n) = m^{-1/2}\left[\frac{1}{m-1} \sum_{i=1}^{m} \left(Z_i(n) - \frac{1}{m} \sum_{j=1}^{m} Z_j(n)\right)^2\right]^{1/2}$$

where

$$Z_i(n) = \int_{(i-1)n/m}^{in/m} Y(s)ds/(n/m)$$

is the i'th batch mean of the process $\{Y(t) : 0 \leq t \leq n\}$. Specializing Theorem 3.9 to our example therefore allows us to conclude that

$$\sqrt{m}\left(\frac{1}{m} \sum_{i=1}^{m} Z_i(n) - \mu\right) \Big/ \left[\frac{1}{m-1} \sum_{i=1}^{m} \left(Z_i(n) - \frac{1}{m} \sum_{j=1}^{n} Z_j(n)\right)^2\right]^{1/2} \Rightarrow t_{m-1}$$

as $n \to \infty$, where t_{m-1} is a Student's-t RV with $m-1$ degrees of freedom. To summarize, we have just shown that the method of batch means, with the number of batches fixed at $m \geq 2$, is asymptotically valid under condition (3.1). This result complements a similar theorem due to BRILLINGER (1973).

$$P\{(\bar{Y}_n(1)-\mu)/g(\bar{Y}_n) \le x\} \to H(x)$$

as $n \to \infty$ for all $x \in \mathbb{R}$. Hence, to obtain a $100(1-\delta)\%$ confidence interval, one selects α and β such that $H(\beta) - H(\alpha) = 1-\delta$ (such α, β exist since $H(\cdot)$ is continuous; also, (3.14) implies that $H(\cdot)$ is strictly increasing). Then, the interval

$$(3.15) \qquad [\bar{Y}_n(1) - g(\bar{Y}_n)\beta,\ \bar{Y}_n(1) - g(\bar{Y}_n)\alpha]$$

is an asymptotic $100(1-\delta)\%$ confidence interval for μ.

The process $(\bar{Y}_n - \mu k)/g(\bar{Y}_n)$ is called a **standardized time series.** Theorem 3.9 and Proposition 3.12 show that every $b \in \mathcal{N}$ gives rise to a particular standardized time series procedure; (3.15) is then the corresponding confidence interval for μ.

4. EXAMPLES OF STANDARDIZED TIME SERIES

Our first example of a standardized time series captures a methodology which has been extensively studied in the simulation literature, namely the method of batch means.

(4.1) EXAMPLE. Let $b_m : C[0,1] \to \mathbb{R}$ be defined by

$$b_m(x) = \left[\left(\frac{m}{m-1}\right) \sum_{i=1}^{m} (x(i/m) - x((i-1)/m))^2\right]^{1/2},$$

for $m \ge 2$. It is easily verified that $b_m \in \mathcal{N}$, so that $g_m = b_m \circ \Gamma \in \mathcal{M}$ (see Proposition 3.12). But

$$(\Gamma(x-\beta k))(t) = x(t) - \beta k(t) - t(x(1) - \beta k(1))$$

$$= x(t) - tx(1) = (\Gamma x)(t)$$

so $\Gamma(x-\beta k) = \Gamma x$; hence, $g(x-\beta k) = g(x)$.

To prove that $\mathcal{M} \subseteq \mathcal{M}^*$, consider $g \in \mathcal{M}$. We claim that g can be represented in the form $g = b \circ \Gamma$, by setting $b = g$. Recall that $g(x) = g(x-\beta k)$ for all $\beta \in \mathbb{R}$. In particular, setting $\beta = x(1)$, we see that $g(x) = g(\Gamma x)$, proving our assertion.

We can now obtain the following result.

(3.13) PROPOSITION. If $g \in \mathcal{M}$, then $B(1)$ is independent of $g(B)$.

PROOF. It is well known that the process $B(t) - tB(1)$ $(0 \leq t \leq 1)$ is independent of $B(1)$ (see p. 84 of [1], for example). In other words, ΓB is independent of $B(1)$, which, of course, implies that $g(B) = (b \circ \Gamma)(B)$ is independent of $B(1)$.

Let $\Phi(x) = P\{B(1) \leq x\}$, $G(x) = P\{g(B) \leq x\}$, and $H(x) = P\{B(1)/g(B) \leq x\}$. Then,

$$H(x) = \int_0^\infty \Phi(xy)\, G(dy) \tag{3.14}$$

by Proposition (3.13). The continuity of $\Phi(\cdot)$ and the bounded convergence theorem imply that the right-hand side of (3.14) is continuous everywhere in x. Thus, by (2.1) and (3.10), it follows that under the conditions of (3.1), where the last equality is due to (3.8ii). These observations immediately yield the theorem.

The proof clearly indicates the role of Assumption (3.8ii); this condition guarantees that $g(X_n)$ does not depend on the unknown parameter μ.

To construct confidence intervals based on (3.10), we need to learn more about the limit RV $B(1)/g(B)$. We start by obtaining an alternative description of $\mathcal{M}$. Let $\Gamma : C[0,1] \to C[0,1]$ be the map define by

$$(\Gamma x)(t) = x(t) - tx(1) .$$

Let $\mathcal{N}$ be the class of functions $b : C[0,1] \to \mathbb{R}$ which satisfy:

(3.11) (i) $b(\alpha x) = \alpha b(x)$ for $\alpha > 0$, $x \in C[0,1]$,

(ii) $P\{(b \circ \Gamma)(B) > 0\} = 1$,

(iii) $P\{B \in D(b \circ \Gamma)\} = 0$.

Set $\mathcal{M}^* = \{g : g = b \circ \Gamma,\ b \in \mathcal{N}\}$.

(3.12) PROPOSITION. $\mathcal{M}^* = \mathcal{M}$.

PROOF. We first show that $\mathcal{M}^* \subseteq \mathcal{M}$. Suppose that $g = b \circ \Gamma$, where $b \in \mathcal{N}$. Clearly, g satisfies (3.8i), (3.8iii), and (3.8iv). For (3.8ii), observe that

(5.8) THEOREM. Suppose $g \in \mathcal{M}$. Then

$$\phi(g) \geq \Phi^{-1}(1-\delta/2)$$

where Φ^{-1} is the inverse of the normal cumulative distribution function Φ.

PROOF. By Lemma 5.7, we may scale g so that

$$z(g;\ 1-\delta/2) = 1 \ .$$

Now, (5.9) implies that

$$H(1) = 1-\delta/2 \ ,$$

or

$$\int_0^\infty \Phi(y)\ G_g(dy) = 1-\delta/2 \ ,$$

where $G_g(dy) = P\{g(B) \in dy\}$ (see (3.12)). Thus, we are to show that

$$\phi(g) = Eg(B) \cdot z(g;\ 1-\delta/2) \tag{5.10}$$

$$= \int_0^\infty (1-\ G_g(y))dy \geq \Phi^{-1}(1-\delta/2) \quad ,$$

subject to

$$\int_0^\infty \Phi(y)\ G_g(dy) = 1-\delta/2 \ . \tag{5.11}$$

Integrating by parts, we find that

$$\int_0^\infty \Phi(y)\, G_g(dy) = [\Phi(y)\, \bar{G}_g(y)]_0^\infty + \int_0^\infty \bar{G}_g(y)\, \phi(y)dy$$

$$= \int_0^\infty \bar{G}_g(y)\, \phi(y)dy \ ,$$

where $\bar{G}_g(y) = 1 - G_g(y)$ and $\phi(y)$ is the normal density function.

Let $K(y)$ be the distribution function defined by

$$K(y) = \begin{cases} 0; & y < p \\ 1; & y \geq p \end{cases}$$

where $p = \Phi^{-1}(1-\delta/2)$. Note that

$$\int_0^\infty \bar{K}(y)\, \phi(y)dy = 1-\delta/2$$

and

$$\int_0^\infty \bar{K}(y)dy = p \ ,$$

where $\bar{K}(y) = 1 - K(y)$. Thus, we can reformulate (5.10) and (5.11) as: show that

(5.12) $$\int_0^\infty (\bar{G}_g(y) - \bar{K}(y))dy \geq 0$$

subject to

(5.13) $$\int_0^\infty (\bar{G}_g(y) - \bar{K}(y))\, \phi(y)dy = 0 \ .$$

Since ϕ is strictly decreasing on $[0,\infty)$, and because

$$\bar{G}_g(y) - \bar{K}(y) \leq 0 \qquad \text{for} \quad y < p \ ,$$

$$\bar{G}_g(y) - \bar{K}(y) \geq 0 \qquad \text{for} \quad y \geq p \ ,$$

it follows that

$$(5.14) \qquad \int_0^p (\bar{G}_g(y) - \bar{K}(y))dy \cdot \phi(p) \geq \int_0^p (\bar{G}_g(y) - \bar{K}(y)) \ \phi(y)dy \ .$$

and

$$(5.15) \qquad \int_p^\infty (\bar{G}_g(y) - \bar{K}(y))dy \cdot \phi(p) \geq \int_p^\infty (\bar{G}_g(y) - \bar{K}(y)) \ \phi(y)dy \ .$$

Adding (5.14) and (5.15) together, we get

$$\int_0^\infty (\bar{G}_g(y) - \bar{K}(y))dy \cdot \phi(p) \geq \int_0^\infty (\bar{G}_g(y) - \bar{K}(y)) \ \phi(y)dy \ .$$

Relation (5.13) then yields (5.12).

(5.16) COROLLARY. Suppose $g \in \mathcal{M}$ is non-negative. Under Assumption (3.1),

$$\varliminf_{n \to \infty} n^{1/2} \, EL_n \geq 2\sigma \, \Phi^{-1}(1-\delta/2) \ .$$

This corollary follows immediately from Propositions 5.1 and 5.3, and Theorem 5.8. The lower bound of Corollary 5.16 has an important interpretation. Consider a steady-state simulation output analysis algorithm which is based on constructing an estimator s_n which consistently estimates σ:

(5.17) $$s_n \Rightarrow \sigma ,$$

as $n \to \infty$. Among the algorithms of this type are the regenerative method of simulation, spectral methods, and autoregressive procedures (see Chapter 3 of BRATLEY, FOX and SCHRAGE (1983) for a description of these techniques). The following proposition is a straightforward application of the converging-together lemma (see p. 25 of [1]).

(5.18) PROPOSITION. If s_n is an estimator satisfying (5.17), then (3.1) implies that

(5.19) $$n^{1/2}(\bar{Y}_n(1) - \mu)/s_n \Rightarrow B(1) ,$$

as $n \to \infty$.

The weak convergence result (5.19) permits construction of asymptotic $100(1-\delta)\%$ confidence intervals for μ:

(5.20) $$\left[\bar{Y}_n(1) - z(\delta) \frac{s_n}{n^{1/2}}, \bar{Y}_n(1) + a(\delta) \frac{s_n}{n^{1/2}}\right] ,$$

where $z(\delta) = \Phi^{-1}(1-\delta/2)$. If L_n is the length of the interval (5.20), it is clear that as $n \to \infty$,

(5.21) $$n^{1/2} L_n \Rightarrow 2\sigma \Phi^{-1}(1-\delta/2) ,$$

which is precisely the lower bound of Corollary 5.16. If $\{s_n; n \geq 1\}$ is uniformly integrable (conditions guaranteeing this appear in GLYNN and IGLEHART (1985b), Section 6), then we further have that

$$\lim_{n \to \infty} n^{1/2} EL_n = 2\sigma\, \Phi^{-1}(1-\delta/2) \quad (5.22)$$

Corollary 5.16, and the limit theorems (5.21) and (5.22) suggest that, from the viewpoint of expected confidence interval length, output analysis methods which consistently estimate σ dominate standardized time series procedures asymptotically.

One further point, pertinent to expected confidence interval length, remains to be investigated. The examples of Section 4 show that for any $k \geq 1$, there exists $g_k \in \mathcal{M}$ such that $B(1)/g_k(B)$ has a Student's-t distribution with k degrees of freedom. If $g_k(X_n)$ is uniformly integrable, then it follows that if $L_n(k)$ is the length of such a confidence interval,

$$\lim_{n \to \infty} n^{1/2} EL_n(k) = 2\sigma\, H_k^{-1}(1-\delta/2) ,$$

where $H_k^{-1}(p)$ is the pth quantile of a Student's-t with k degrees of freedom. Since

$$\lim_{k \to \infty} H_k^{-1}(1-\delta/2) = \Phi^{-1}(1-\delta/2) ,$$

this discussion suggests that

$$\inf_{g \in \mathcal{M}} \varliminf_{n \to \infty} n^{1/2} EL_n = 2\sigma\, \Phi^{-1}(1-\delta/2) ; \quad (5.23)$$

thus, the lower bound of Corollary 5.16 is tight. Relation (5.23) raises the question of whether there exists $g \in \mathcal{M}$ such that

$$\text{(5.24)} \qquad \varliminf_{n \to \infty} n^{1/2} EL_n = 2\sigma \, \Phi^{-1}(1-\delta/2) \ ;$$

in other words, is the lower bound attained within $\mathcal{M}$?

A glance at the proof of Theorem 5.8 shows that

$$\phi(g) > \Phi^{-1}(1-\delta/2)$$

unless $G_g(dy)$ is a point-mass distribution. Thus, in order to find $g \in \mathcal{M}$ satisfying (5.24), it must be that

$$\text{(5.25)} \qquad P\{g(\sigma B) = \alpha\sigma\} = 1$$

for some $\alpha > 0$. Our next result shows that such a g cannot exist.

(5.26) PROPOSITION. There exists no $g \in \mathcal{M}$ such that (5.25) holds.

PROOF. We will prove something stronger: the requirements $P\{B \in D(g)\} = 0$ and (5.25) are incompatible. We start by showing that for every $x \in C_0[0,1] \equiv \{x \in C[0,1] : x(0) = 0\}$ and $\varepsilon > 0$,

$$\text{(5.27)} \qquad P\{\rho(\sigma B, x) < \varepsilon\} > 0 \ .$$

To see this, fix $x \in C_0[0,1]$ and $\delta > 0$. Since $[0,1]$ is compact, x is uniformly continuous on $[0,1]$, so there exists $N = N(\varepsilon)$ such that

$$(5.28)\qquad |x(t) - x(k/N)| < \varepsilon/4$$

for $kN \le t \le (k+1)/N$, where $0 \le k < N$. Now, the independent increments of Brownian motion imply that if $\Delta z(k/N) \equiv z((k+1)/N) - z(k/N)$, then

$$(5.29)\quad P(A(\varepsilon)) \equiv P\Big\{|\sigma\Delta B(k/N) - \Delta x(k/N)| < \varepsilon/4N,\quad \sigma \max_{k/N \le t \le (k+1)/N} |B(t) - B(k/N)| < \varepsilon/2,\quad 0 \le k < N\Big\}$$

$$= \prod_{k=0}^{N-1} P\Big\{|\sigma\Delta B(k/N) - \Delta x(k/N)| < \varepsilon/4N,\quad \sigma \max_{k/N \le t \le (k+1)/N} |B(t) - B(k/N)| < \varepsilon/2\Big\} > 0 ,$$

by virtue of the fact that for any z with $|z| < \eta$, $P\{|B(t) - z| < \eta$, $\max_{0 \le s \le t} |B(s)| < 2\eta\} > 0$. Now, on the event $A(\varepsilon)$, a simple triangle inequality argument shows that

$$|\sigma B(t) - x(t)| < \varepsilon$$

for $0 \le t \le 1$ (use (5.28)), proving (5.27).

From (5.27) and (5.25), it follows that for some $\alpha > 0$

$$P\{\rho(\sigma B, x) < \varepsilon,\ g(\sigma B) = \alpha\sigma\} > 0 ,$$

so that there necessarily exists $y = y(x,\varepsilon)$ such that $\rho(y,x) < \varepsilon$ with $g(y) = \alpha\sigma$. Thus, the range of g over any ε-neighborhood of x contains the set $\{\alpha\sigma\colon \sigma > 0\}$; clearly, then g can not be continuous at x.

Hence, $x \in D(g)$. Since x was arbitrary, this implies that $D(g) = C_0[0,1]$, violating the assumption $P\{B \in D(g)\} = 0$.

We now turn to the question of end-point variability. To be precise, observe that if $g \in \mathcal{M}$, then (3.1) implies that

$$n^{1/2} L_n \Rightarrow \sigma g(B) \cdot (\beta - \alpha) \tag{5.30}$$

as $n \to \infty$ (see (5.2) for a more complete argument). The limit distribution of the confidence interval length is, of course, degenerate if (and *only if*) $g(B)$ is degenerate. Suppose that, in fact, $g(B)$ is degenerate so that there exists α such that $P\{g(B) = \alpha\} = 1$. Note that $\alpha > 0$ by (3.8iii). On the other hand, it follows from (3.8i) that

$$P\{g(\sigma B) = \alpha\sigma\} = 1 \tag{5.31}$$

for all $\sigma > 0$. But (5.31) is, of course, just (5.25); Proposition 5.26 therefore proves that no such g can exist. Consequently, we may conclude that $g(B)$ must be non-degenerate. The limit theorem (5.30) therefore states that L_n exhibits non-degenerate random fluctuations of order $n^{-1/2}$.

Another way to quantify the above phenomenon is to examine the quantity $E(L_n - EL_n)^2$.

(5.32) PROPOSITION. If $\{g^2(X_n): n \geq 1\}$ is uniformly integrable, then under (3.1),

$$\lim_{n \to \infty} nE(L_n - EL_n)^2 = \sigma^2 E(g(B) - Eg(B))^2 \, (\beta-\alpha)^2 \tag{5.33}$$

provided $g \in \mathcal{M}$. Furthermore, the right-hand side of (5.33) is positive.

PROOF. The uniform integrability of $\{g^2(X_n): n \geq 1\}$ implies that of $\{g(X_n): n \geq 1\}$ (see p. 100 of [4]), so

$$\lim_{n \to \infty} n^{1/2} Eg(\bar{Y}_n) = \lim_{n \to \infty} Eg(X_n) = Eg(B) ,$$

and

$$\lim_{n \to \infty} n \, Eg^2(\bar{Y}_n) = \lim_{n \to \infty} Eg^2(X_n) = Eg^2(B) ;$$

combining the above two limit relations yields (5.33). As for the positivity, this follows from the non-degeneracy of $g(B)$ for $g \in \mathcal{M}$.

We now wish to compare the end-point variability of standardized time series procedures to that obtained via methods which consistently estimate σ. Our analysis will be restricted to the regenerative method of simulation; we do this only because the required limit theorems are available in this context.

As (5.21) indicates, $n^{1/2} L_n$ converges to a degenerate r.v. Thus, L_n asymptotically exhibits no random fluctuations of order $n^{-1/2}$. We can, in fact, be more precise.

(5.34) PROPOSITION. Let Y be a regenerative process satisfying $E((Y_1|f|)^9 + \tau_1^9) < \infty$ (see (3.3) for the definition of $Y_n(|f|)$ and τ_n). Then, if s_n is the regenerative estimator for σ, there exists η such that

(5.35) (i) $n(L_n - EL_n) \Rightarrow \eta N(0,1)$ as $n \to \infty$;

(ii) $n^2 E(L_n - EL_n)^2 \to \eta^2$ as $n \to \infty$.

PROOF. Under the above moment hypothesis, there exists κ such that

$$n^{1/2}(s_n - \sigma) \Rightarrow \kappa N(0,1) \, , \tag{5.36}$$

as $n \to \infty$; furthermore, the sequence $\{n(s_n-\sigma)^2 : n \geq 1\}$ is uniformly integrable (see Sections 5 and 6 of [7]). Thus,

$$nE\left(L_n - \frac{2z(\delta)\sigma}{n^{1/2}}\right) \to 0 \, , \tag{5.37}$$

as $n \to \infty$; combining (5.36) and (5.37), we get (5.35i). For (ii), we use the uniform integrability to obtain

$$nE(s_n - \sigma) \to \kappa^2 \, ;$$

this evidently implies that

$$n^2 E(L_n - EL_n)^2 \to 4z^2(\delta)\, \kappa^2 \, ,$$

proving (ii).

We conclude that the end-point variability of the regenerative confidence interval is of order n^{-1}, as opposed to $n^{-1/2}$ for standardized time series.

6. SUMMARY

Building on work of SCHRUBEN, we have developed a general framework for the analysis of standardized time series. Under mild assumptions on the output process (see (3.1)), the method of standardized time series produces asymptotically valid confidence intevals for steady-state parameters. However, these intervals are asymptotically larger (see (5.16)) and more variable (see (5.33)) then those steady-state intervals obtained by a method which consistently estimates the appropriate steady-state variance constant (such as the regenerateive method). In this sense, standardized time series confidence intervals are asymptotically less desirable then those constructed by a consistent estimation.

These results do not, however, preclude the possibility that standardized times series may be superior in certain small sample context; this remains an area for future work.

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